

Holographic Flow of Anomalous Transport Coefficients

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Abstract

We study the holographic flow of anomalous conductivities induced by gauge and gravitational Chern-Simons terms. We find that the contribution from the gauge Chern-Simons term gives rise to a flow that can be interpreted in terms of an effective, cutoff dependent chemical potential. In contrast the contribution of the gauge-gravitational Chern-Simons term is just the temperature squared and does not flow.

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1 Introduction

Anomalies appear in the context of relativistic quantum field theories. In four dimensions chiral anomalies [1] involve triangle diagrams with either only vector currents or vector currents and the energy momentum tensor, in which case one speaks of a (mixed gauge-) gravitational anomaly [2]. They are responsible for the breakdown of a classical symmetry due to quantum effects. If the symmetry is local anomalies impose severe restrictions on the structure and definition of gauge theories (for comprehensive reviews on anomalies see [3]). In the case of a symmetry generated by T_A , and considering only right-handed fermions, the presence of a chiral anomaly in vacuum is encoded in a non-vanishing $d_{ABC} = \frac{1}{2}\text{tr}(T_A\{T_B, T_C\})$. The corresponding parameter in the case of the gravitational anomaly is $b_A = \text{tr}(T_A)$.

Recently, it has been pointed out that at finite temperature and density, anomalies are responsible for the appearance of new non-dissipative transport phenomena [4, 5, 6]. In the chiral magnetic effect an external magnetic field induces a current parallel to it

$$J^\mu = \sigma_B B^\mu \quad (1)$$

where σ_B is the chiral magnetic conductivity and $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}u_\nu F_{\rho\lambda}$. A second effect is the chiral vortical effect. It refers to the creation of a current parallel to vortices in the fluid

$$J^\mu = \sigma_V \omega^\mu \quad (2)$$

with $\omega^\mu = \epsilon^{\mu\nu\rho\lambda}u_\nu\partial_\rho u_\lambda$ being the vorticity vector and u_μ the fluid four-velocity.

The contribution of the gravitational anomaly to these transport coefficients was first obtained in a weakly coupled gas of chiral fermions in [7]. A holographic model that confirmed these findings at strongly coupling was developed and studied in [8].

The chiral magnetic and the chiral vortical conductivities can be calculated from first principles via the Kubo formulae [9] (Latin letters denote purely spatial indices)

$$\sigma_B = \lim_{p_n \rightarrow 0} \frac{i}{2p_c} \sum_{a,b} \epsilon_{abc} \langle J^a J^b \rangle (\omega = 0, \vec{p}) \quad (3)$$

$$\sigma_V = \lim_{p_n \rightarrow 0} \frac{i}{2p_c} \sum_{a,b} \epsilon_{abc} \langle J^a T_0^b \rangle (\omega = 0, \vec{p}) \quad (4)$$

There are related transport coefficients for the energy current $T^{0\mu}$

$$T^{0\mu} = \sigma_B^\epsilon B^\mu, \quad (5)$$

$$T^{0\mu} = \sigma_V^\epsilon \omega^\mu. \quad (6)$$

They are calculated via the Kubo formulae

$$\sigma_B^\epsilon = \lim_{p_n \rightarrow 0} \frac{i}{2p_c} \sum_{a,b} \epsilon_{abc} \langle T_0^a J^b \rangle (\omega = 0, \vec{p}) \quad (7)$$

$$\sigma_V^\epsilon = \lim_{p_n \rightarrow 0} \frac{i}{2p_c} \sum_{a,b} \epsilon_{abc} \langle T_0^a T_0^b \rangle (\omega = 0, \vec{p}) \quad (8)$$

The role played by the gravitational anomaly was further studied in [10], where an ideal Weyl gas in arbitrary even dimensions was considered. This leads to a generalization of the anomalous conductivities valid for any (even) dimension and an expression that relates the anomaly induced transport coefficients to the anomaly polynomial of the Ideal Weyl gas. Furthermore, in [11], a definition for the local entropy current for higher-curvature gravitational theories was proposed and the Fluid/Gravity correspondence was applied to compute the first order conductivities in the presence of the gravitational anomaly.

Within the gauge-gravity duality the running with the holographic coordinate can be interpreted as a type of renormalization group (RG) flow in the dual field theory [12]. The first application of this holographic flow to transport coefficients is [13] where it was shown that the electric conductivity and the shear viscosity have a trivial flow. It is known now that some of the transport coefficients indeed present a non-trivial flow (see, for instance [14]). The extension to finite chemical potential has been studied in [15]. Recently, there is a renewed interest in this subject due to the explicit holographic construction of the Wilsonian Renormalization Group [16], which has made possible to show that multi-trace deformations in the effective action are induced after integrating out high energy modes. Finally, in [17] it was pointed out that all the apparently different frameworks used over time to study the holographic flow are actually equivalent.

It is natural to analyze the holographic flow of the anomalous conductivities as well. In this paper we present several approaches to compute the different flows and show that all the methods lead to the same results as expected. In Section 2 the setup is presented. We interpret the holographic flow, defined as in [13], as a cutoff flow that arises by varying the holographic cutoff at finite holographic coordinate value $r = \Lambda$. Section 3 studies the case of gauge fields without taking into account the backreaction onto metric perturbations. Then we generalize the previous approach to include also the metric perturbations and present a non-covariant method to calculate the flow equations for the retarded Green's functions. Somewhat surprisingly we do find a non-trivial flow but give it a natural interpretation as a cutoff flow of an effective chemical potential. The flow of the correlators is computed by explicitly solving the equations of motion for a system restricted to live between the black brane horizon and a hyper surface placed at finite $r = \Lambda$, which acts as a boundary. The section ends with a discussion regarding the compatibility of the results so obtained with the flow equations. It is shown that both approaches are in agreement. In Section 4 the attention is focused on the gravitational anomaly. We discuss subtleties concerning the definition of a Dirichlet problem and the necessity for the inclusion of a boundary term to ensure that

the correct form for the divergence of the current is found. Contrary to what happens for the chemical potential, we find that the temperature term stemming from the gravitational anomaly does not flow.

We conclude in Section 5 with a discussion.

2 Setup

Lets show how transport coefficients flow with a variation of the holographic cutoff scale. We define the theory with a cutoff as:

$$S = \frac{1}{e^2} \int_{r < \Lambda} \sqrt{-g} \left(-\frac{1}{4} F_{MN} F^{MN} \right) \quad (9)$$

We consider this theory in a general black brane background of the form

$$ds^2 = -g_{tt} dt^2 + g_{rr} dr^2 + g_{ii} d(x^i)^2. \quad (10)$$

We assume that the above metric has an event horizon at $r = r_H$ and that every component depends only on r . The boundary is placed at $r = \Lambda$. The metric is also assumed to be regular except at the horizon and possibly in the limit $\Lambda \rightarrow \infty$. The current of the holographic dual field theory is

$$J^\mu = \frac{1}{e^2} \sqrt{-g} F^{\mu r} \Big|_{r=\Lambda}. \quad (11)$$

In the gauge $A_r = 0$ the x -component of its variation due to a small perturbation of the gauge field reads

$$J^x = \frac{-1}{e^2} \sqrt{-g} g^{xx} g^{rr} \dot{a}(x, r) \Big|_{r=\Lambda} \quad (12)$$

where $\dot{a} = da/dr$ is the r -derivative of the aforementioned perturbation.

We define $a(x, r)$ as $a(x, r) = \frac{a(r)}{a(\Lambda)} a^{(0)}(x)$, so that it is normalized at the boundary to $a(x, \Lambda) = a^{(0)}(x)$ and $a(r)$ solves the radial wave equation

$$\ddot{a}(r) + \frac{1}{2} \dot{a}(r) (g^{tt} \dot{g}_{tt} + g^{xx} \dot{g}_{xx} - g^{rr} \dot{g}_{rr}) + g_{rr} (\omega^2 g^{tt} - k^2 g^{xx}) a(r) = 0 \quad (13)$$

On the other hand, we *define* the electric conductivity at the boundary as $J^x = \sigma_E(\Lambda) E(\Lambda)$, where $E(\Lambda) = -i\omega a^{(0)}$ is the external applied electric field. Comparing this to equation (12) we conclude

$$\sigma_E(\Lambda) = \frac{-i}{e^2\omega} \sqrt{-g} g^{xx} g^{rr} \frac{\dot{a}(r)}{a(\Lambda)} \Big|_{r=\Lambda} \quad (14)$$

Varying the cutoff $\Lambda \rightarrow \Lambda + d\Lambda$ we find the differential of the electric conductivity

$$\frac{d\sigma_E(\Lambda)}{d\Lambda} = \frac{-i}{e^2\omega} \left[\frac{d}{dr} \left(\sqrt{-g} g^{xx} g^{rr} \frac{da(r)/dr}{a(r)} \right) \right]_{r=\Lambda} \quad (15)$$

This equation shows that we can study the flow of the transport coefficients with the cutoff reformulating it as the evolution with respect to the coordinate r , by formally identifying r with Λ .

We can use now the equation of motion for the perturbation $a(r)$ and the definition of the conductivity (14) to derive the flow equation

$$\frac{d\sigma_E(\omega, k)}{d\Lambda} = -i\omega \left[\frac{e^2}{\sqrt{-g}} g_{rr} g_{xx} \sigma_E^2 + \frac{\sqrt{-g}}{e^2} g^{xx} \left(g^{tt} + \frac{k^2}{\omega^2} g^{xx} \right) \right] \quad (16)$$

This the flow equation first derived in [13]. It can be solved by demanding infalling boundary conditions on the horizon. In particular the flow for the DC conductivity turns out to be trivial $\dot{\sigma}_E = 0$. In this case the electric conductivity is completely determined by its value on the horizon via the membrane paradigm

$$\sigma_E(\Lambda) = \sigma_E(r_H) = \frac{1}{e^2}. \quad (17)$$

3 Flow of anomalous conductivities

We will apply now the strategy outlined before to the anomalous transport coefficients. Two models will be considered. First we discuss a model in which we neglect the backreaction of the gauge field fluctuations on the metric. We will study the interplay between two $U(1)$ symmetries which we call vector and axial ones. This allows to model the chiral magnetic and the chiral separation effect. A second model will use only one anomalous $U(1)$ symmetry but we will also include the backreaction onto the metric. This allows to study also the flow of the chiral vortical conductivity and the flow of the anomalous transport coefficients related to the energy current.

3.1 Vector and Axial symmetries

We will apply the aforementioned strategy to the chiral magnetic conductivity [18]. Its proper definition requires the interplay between a vector like $U(1)$ symmetry and an axial $U(1)$ symmetry. Holographic models have been investigated in [19, 20, 21]. The model allows for the definition of the chiral magnetic conductivity and axial conductivities involving external axial magnetic fields. Its action is given by [21]

$$S = \int \sqrt{-g} \left(-\frac{1}{4g_V^2} F_{MN}^V F_V^{MN} - \frac{1}{4g_A^2} F_{MN}^A F_A^{MN} + \frac{\kappa}{2} \epsilon^{MNPQR} A_M (F_{NP}^A F_{QR}^A + 3F_{NP}^V F_{QR}^V) \right) \quad (18)$$

where V stands for 'vector' and A for 'axial'. The Lagrangian contains two Maxwell actions for vector and axial gauge fields and a particular choice of Chern-Simons term. In what follows, we will stick to the notation of [21]; concretely, we define the epsilon symbol as $\epsilon(ABCDE) = -\sqrt{-g}\epsilon^{ABCDE}$, with $\epsilon(rxyz) = 1$ (r corresponds to the fifth coordinate).

From the boundary term of this action, after perturbing both the axial and the vector gauge fields, we obtain an expression for the boundary theory currents

$$J^\mu = \left(\frac{1}{g_V^2} \sqrt{-g} F_V^{\mu r} + 6\kappa \epsilon^{\mu\nu\rho\lambda} A_\nu F_{\rho\lambda}^V \right) \Big|_{r=\Lambda}, \quad (19)$$

$$J_5^\mu = \left(\frac{1}{g_A^2} \sqrt{-g} F_A^{\mu r} + 2\kappa \epsilon^{\mu\nu\rho\lambda} A_\nu F_{\rho\lambda}^A \right) \Big|_{r=\Lambda}, \quad (20)$$

where $\epsilon^{\mu\nu\rho\lambda} \equiv \epsilon^{r\mu\nu\rho\lambda}$. The coefficients in front of the Chern-Simons terms are crucial to ensure that the vector current is non-anomalous $D_\mu J_V^\mu = 0$. The axial current is anomalous $D_\mu J_5^\mu = -\frac{\kappa}{2} \epsilon^{\mu\nu\rho\lambda} (3F_{\mu\nu}^V F_{\rho\lambda}^V + F_{\mu\nu}^A F_{\rho\lambda}^A)$ [21]. Comparing with the standard result from the one loop triangle calculation, we find $\kappa = -\frac{N_c}{24\pi^2}$ for a dual strongly coupled $SU(N_c)$ gauge theory for a mass less Dirac fermion in the fundamental representation. Note also that both currents are invariant under vector gauge transformations but not under axial gauge transformations.

The equations of motion for the gauge fields are

$$\frac{1}{g_A^2} \nabla_N F_A^{NM} + \frac{3\kappa}{2} \epsilon^{MNPQR} (F_{NP}^A F_{QR}^A + F_{NP}^V F_{QR}^V) = 0 \quad (21)$$

$$\frac{1}{g_V^2} \nabla_N F_V^{NM} + 3\kappa \epsilon^{MNPQR} (F_{NP}^A F_{QR}^V) = 0 \quad (22)$$

In order to study the flow of the conductivities with the fifth coordinate, we will proceed as follows:

- We introduce an axial and vector perturbation of the gauge fields

$$A_M = A_M^{(0)} + a_M(y, t, r) \quad (23)$$

$$V_M = V_M^{(0)} + v_M(y, t, r) \quad (24)$$

We switch on perturbations only in the z and x -directions (transverse directions): $a_z(y, t, r)$, $v_z(y, t, r)$, $a_x(y, t, r)$, $v_x(y, t, r)$

- Since the Chern-Simons contribution to the current depends only on the intrinsic gauge fields on the cutoff surface, its flow is trivial. The non-trivial part of a possible flow is completely contained in the covariant currents

$$J^{(1)x} = \left(\frac{1}{g_V^2} \sqrt{-g} F_V^{(1)xr} \right) \Big|_{r=\Lambda} \quad (25)$$

$$J_5^{(1)x} = \left(\frac{1}{g_A^2} \sqrt{-g} F_A^{(1)xr} \right) \Big|_{r=\Lambda} \quad (26)$$

- We define our transport coefficients as the response to the perturbations and in terms of the previously defined covariant currents as

$$J^{(1)x} = \sigma_{CME} \epsilon(rtxyz) F_{yz}^{(1)V} + \sigma_{axial} \epsilon(rtxyz) F_{yz}^{(1)A}, \quad (27)$$

$$J_5^{(1)x} = \sigma_{axial} \epsilon(rtxyz) F_{yz}^{(1)V} + \sigma_{55} \epsilon(rtxyz) F_{yz}^{(1)A}, \quad (28)$$

σ_{axial} defines the vector current generated by an external axial magnetic field. Observe that, in order not to have $F_{\{A,V\}}^{(1)xr} = 0$ identically, one has to turn on the perturbations $a_x(y, t, r), v_x(y, t, r)$. However, these do not play a role when studying the flow of the anomalous conductivities for they induce contributions that tend to zero in the low ω , low k limit, very much as occurs in [13].

The value of the background gauge fields is [21]

$$A_0^{(0)} = \alpha - \frac{\mu_5 r_H^2}{r^2} \quad (29)$$

$$V_0^{(0)} = \gamma - \frac{\mu r_H^2}{r^2} \quad (30)$$

The integration constants α and γ can be fixed by e.g. demanding that the gauge fields vanish on the horizon. In any case the covariant currents do not depend on these integration constants. The consistent currents (19), (20) do however depend on them through the Chern-Simons currents. For a discussion of this dependence see [20, 21].

The procedure consists of using the equations of motion to find the value of $\partial_r \sigma$, where σ a generic conductivity defined at some hyper surface $r = \Lambda$. In fact, we only need the equations of motion projected onto x and the Bianchi identity associated with the indices (r, y, z) to obtain an expression of the derivative with respect to r of the different transport coefficients. From the simple form of our metric it can be seen that the vector normal to a hyper surface of $x=\text{constant}$ reads $n_\mu^x = \sqrt{g_{xx}}(0, 0, 1, 0, 0)$. Hence, for the vector gauge field we have

$$n_M^x \left[\frac{1}{g_V^2} \nabla_N F_V^{NM} + 3\kappa \epsilon^{MNPQR} (F_{NP}^A F_{QR}^V) \right] = 0 \quad (31)$$

Taking advantage of the relation $\nabla_N F^{NM} = \frac{1}{\sqrt{-g}} \partial_N (\sqrt{-g} F^{NM})$ and the definition of the currents (25) and (26), we arrive at

$$\partial_r J^{(1)x} = -12\kappa \sqrt{-g} \epsilon^{rtxyz} \left(F_{tr}^{A(0)} F_{yz}^{V(1)} + F_{yz}^{A(1)} F_{tr}^{V(0)} \right) \quad (32)$$

where we have neglected $F_V^{(1)tx}; F_V^{(1)yx}$ for these modes lead to vanishing contributions in the low momentum and low frequency limit, as mentioned before. Besides, we have carried out the contraction $\epsilon^{xNPQR} F_{NP}^A F_{QR}^V = -4\epsilon^{rtxyz} (F_{tr}^A F_{yz}^V + F_{yt}^A F_{rz}^V + F_{tz}^A F_{ry}^V + (A \leftrightarrow V))$. The Bianchi identity to first order associated with indices (r, y, z) reads

$$\partial_r F_{yz}^{\{V,A\}(1)} + \partial_y F_{zr}^{\{V,A\}(1)} + \partial_z F_{ry}^{\{V,A\}(1)} = 0 \quad (33)$$

Assuming $\partial_z F_{ry}^{\{V,A\}(1)} \sim g_{rr} g_{yy} \partial_z J^{y(1)} = 0$ we obtain

$$\partial_r F_{yz}^{\{V,A\}(1)} = -\frac{g_{zz} g_{rr}}{\sqrt{-g}} g_{\{V,A\}}^2 \partial_y J_{\{V,A\}}^{(1)z} \quad (34)$$

Now, making use of these ingredients, the computation of $\partial_r \sigma$ is immediate:

$$\partial_r \sigma_{CME} = \lim_{\omega, k \rightarrow 0} \left[\frac{\partial_r J^{x(1)}}{\epsilon(rtxyz) F_{yz}^{V(1)}} - \frac{J^{x(1)}}{(\epsilon(rtxyz) F_{yz}^{V(1)})^2} \partial_r F_{yz}^{V(1)} \right]_{a_M=0} \quad (35)$$

Plugging (32) and (34) into (35) we find, in momentum space

$$\partial_r \sigma_{CME} = \lim_{\omega, k \rightarrow 0} \left[12\kappa F_{tr}^{A(0)} + ik \sigma_{CME} \frac{g_{zz} g_{rr}}{\sqrt{-g}} g_V^2 \frac{J^{(1)z}}{\epsilon(rtxyz) F_{yz}^{V(1)}} \right] \quad (36)$$

Taking the limit $\omega, k \rightarrow 0$ and substituting $F_{tr}^{V(0)} = -\partial_r A_0^{(0)} = -2\frac{\mu_5 r_H^2}{r^3}$, we get the following flow equation for the chiral magnetic conductivity

$$\partial_r \sigma_{CME} = -24\kappa \frac{\mu_5 r_H}{r^3} \quad (37)$$

whose solution is

$$\sigma_{CME} = C + 12\kappa \frac{\mu_5 r_H^2}{r^2}. \quad (38)$$

C is an integration constant that we must fix. In order to do that, we impose in-falling boundary conditions for the perturbations (or, equivalently, regularity at the horizon

$r = r_H$ [21]). This in turn implies that the fields must depend only on the combination $dv = dt + \sqrt{\frac{g_{rr}}{g_{tt}}} dr$ [13]. Therefore, in the $A_r = 0$ gauge, we have

$$\partial_r A_x = \sqrt{\frac{g_{rr}}{g_{tt}}} \partial_t A_x \text{ at } r = r_H \quad (39)$$

This condition forces directly $J^{(1)x}(r = r_H)$ to be

$$j^{(1)x}(r = r_H) \sim F^{(1)xr}(r = r_H) \sim E^i \quad (40)$$

Imposing these infalling boundary conditions results therefore in a vanishing chiral magnetic conductivity at the horizon for the covariant current ¹ Thus the integration constant C can be fixed simply by the condition

$$\sigma_{CME}(r = r_H) = 0 \rightarrow C = -12\kappa\mu_5 = \frac{N_c\mu_5}{2\pi^2} \quad (41)$$

(Recall that $\kappa = -\frac{N_c}{24\pi^2}$).

The transport coefficients (27) and (28) can be calculated in an analogous way. For the axial current the projected equation is

$$n_{xM} \left[\frac{1}{g_A^2} \nabla_N F_A^{NM} + \frac{3\kappa}{2} \epsilon^{MNPQR} (F_{NP}^A F_{QR}^A + F_{NP}^V F_{QR}^V) \right] = 0 \quad (42)$$

which implies

$$\partial_r j_5^{(1)x} = \sqrt{-g} 12\kappa \epsilon^{rtxyz} \left(\frac{2\mu_5}{r^3} F_{yz}^{(1)A} + \frac{2\mu}{r^3} F_{yz}^{(1)V} \right) \quad (43)$$

The values of the conductivities at $r = \Lambda$ then read

$$\sigma_{CME}(\Lambda) = \frac{N_c\mu_5}{2\pi^2} \left(1 - \frac{r_H^2}{\Lambda^2} \right) \quad (44)$$

$$\sigma_{axial}(\Lambda) = \frac{N_c\mu}{2\pi^2} \left(1 - \frac{r_H^2}{\Lambda^2} \right) \quad (45)$$

$$\sigma_{55}(\Lambda) = \frac{N_c\mu_5}{2\pi^2} \left(1 - \frac{r_H^2}{\Lambda^2} \right) \quad (46)$$

¹Note that the consistent currents might have non-vanishing chiral magnetic conductivity on the horizon due to the Chern-Simons contribution and depending on the value of the integration constant α .

As expected, the result in the limit $r \rightarrow \infty$ is precisely the one obtained in [21] using AdS/CFT techniques.

In view of the topological nature and the non-renormalization theorem for the chiral magnetic conductivity it is at first sight somewhat surprising to find a non-trivial flow. This flow becomes however natural if we define the chemical potential in its elementary way as the energy needed to introduce one unit of charge into the ensemble. In the holographic dual this corresponds to bring a unit of charge from the boundary, now situated at $r = \Lambda$ behind the horizon. The energy difference between a unit of charge at the boundary and a unit of charge at the horizon is just given by $A_0(\Lambda) - A_0(r_H) = \mu(\Lambda)$. This defines an effective chemical potential in the theory equipped with the cutoff Λ . In fact the definition of such an effective chemical potential is natural even in field theory. If we have a momentum cutoff of order Λ we can localize a unit charge only inside a volume within a radius of order $1/\Lambda$. Thermalizing this unit of charge means spreading it out over the entire ensemble. The difference in energy between the two configurations, the unit of charge localized within $1/\Lambda$ and spread out over the ensemble again is the effective chemical potential.

All the anomalous conductivities can therefore be expressed in the form

$$\sigma(\Lambda) = \frac{N_c \mu(\Lambda)}{2\pi^2}. \quad (47)$$

They are linear in the chemical potential and the numerical coefficient is independent of the cutoff. In this sense they obey the expected non-renormalization theorem.

3.2 Inclusion of metric perturbations

In this section we compute the flow equations for the Green's functions associated with generic response. The method can be described as follows: we need to consider two equations. One is the constitutive equation

$$\langle \mathcal{O}_j \rangle = \sum_i^N G_j^i \phi_i \quad (48)$$

and the other one is the covariant holographic definition of the one-point functions, evaluated on some perturbed state. Generically, these would be a functional of the perturbations and its derivatives (the dot means d/dr . We will be using both notations indistinctly).

$$\langle \mathcal{O}_j \rangle = - \sum_i^N (\mathcal{F}_j^i \phi_i + \mathcal{H}_j^i \dot{\phi}_i) \quad (49)$$

Taking the r -derivative in both equations, we can force them to be equal. Observe that, from (49), we expect terms containing $\mathcal{H}_j^i \ddot{\phi}_i$. After using the equations of motion, we will be left with some expression involving only ϕ and $\dot{\phi}$. Then, by equating (48) and (49), it

is possible to find a formula for $\dot{\phi}_j = \sum_i^N K_j^i \phi_i$ so that eventually we are able to write the r-derivative of (49) as an expansion in the perturbations only.

On the other hand, differentiating (48) and using again $\dot{\phi}_j = \sum_i^N K_j^i \phi_i$, we are lead to an expression in terms of G_j^i , \dot{G}_j^i and ϕ_i .

Imposing that the r-derivative of (49) and that of (48) are identical, we finally arrive at

$$0 = \sum_i^N A_j^i \phi_i \quad (50)$$

where A_j^i is a functional of G_j^i and their first derivatives. Assuming now that the different perturbations are independent from each other, we get N independent equations

$$A_j^i = 0 \quad (51)$$

which are nothing but differential equations for G_j^i . Remarkable enough, the flow equations for the retarded correlators are of first order in r-derivatives.

3.2.1 Application to the anomalous conductivities

In what follows we will derive the flow equations in the presence of a pure gauge Chern-Simons term (no gravitational anomaly) by using the procedure detailed in the previous section. The model reads

$$S = S_{EH} + S_{GH} + \frac{1}{16\pi G} \int_{r < \Lambda} \sqrt{-g} \left(-\frac{1}{4} F_{MN} F^{MN} + \frac{\kappa}{3} \epsilon^{MNPQR} A_M F_{NP} F_{QR} \right) \quad (52)$$

where S_{EH} denotes the Einstein-Hilbert action with negative cosmological constant and S_{GH} is the Gibbons-Hawking term on the boundary $r = \Lambda$. The Chern-Simons coupling is here related to the anomaly for a single chiral fermion by $\kappa = -G/(2\pi)$.

Since we need now the precise equations of motion for the metric fluctuations we will specialize the analysis to a Reissner-Nordstrom AdS Black Brane

$$ds^2 = \frac{r^2}{L^2} (-f(r) dt^2 + d\vec{x}^2) + \frac{L^2}{r^2 f(r)} dr^2 \quad (53)$$

$$A^{(0)} = \phi(r) dt = -\frac{\mu r_H^2}{r^2} \quad (54)$$

The integration constant in the gauge field is set that it vanishes for $r \rightarrow \infty$. The horizon of the black hole is located at $r = r_H$ and the blackening factor is $f(r) = 1 - \frac{ML^2}{r^4} + \frac{Q^2 L^2}{r^6}$. The parameters M, Q are related to the chemical potential at infinity μ and r_H by $M = \frac{r_H^4}{L^2} + \frac{Q^2}{r_H^2}$, $Q = \frac{\mu r_H^2}{\sqrt{3}}$. Finally, the Hawking temperature is given by

$$T = \frac{r_H^2}{4\pi L^2} \dot{f}(r_H) = \frac{2r_H^2 M - 3Q^2}{2\pi r_H^5} \quad (55)$$

In what follows, we consider perturbations of momentum k in the y -direction at zero frequency. It is only necessary to turn on the shear sector, that is, the perturbations are written as a_α , h_t^α , where $\alpha = x, z$ ². It is more convenient to work with the coordinate $u = \frac{r_H^2}{r^2}$ instead of r .

The equations of motion for the perturbations derived from (52), when $\omega = 0$ and to $\mathcal{O}(k)$, read

$$0 = B_\alpha''(u) + \frac{f'(u)}{f(u)} B_\alpha'(u) - \frac{h_t^{\alpha'}(u)}{f(u)} + ik\epsilon_{\alpha\beta}\bar{\kappa} \frac{B_\beta(u)}{f(u)}, \quad (56)$$

$$0 = h_t^{\alpha''}(u) - \frac{h_t^{\alpha'}(u)}{u} - 3auB_\alpha'(u) \quad (57)$$

where $\bar{\kappa} = \frac{4\mu\kappa L^3}{r_H^2}$.

The operators that we will be working with have the following form when evaluated in a perturbed state (for further details see [9])

$$\delta J^\alpha = \frac{r_H^2}{8\pi GL^3} (f(u)a'_\alpha - \mu h_t^\alpha) \quad (58)$$

$$\delta t_t^\alpha = \frac{r_H^4 f(u)}{8\pi GL^5 u} \left(h_t^{\prime\alpha} - \frac{3}{u} h_t^\alpha \right) \quad (59)$$

where the prime stands for d/du . Differentiating (58) and (59) we are left with

$$(\delta J^\alpha)' = \frac{r_H^2}{8\pi GL^3} (a_\alpha''(u)f(u) + a'_\alpha(u)f'(u) - \mu h_t^{\prime\alpha}) \quad (60)$$

$$(\delta t_t^\alpha)' = \frac{r_H^4 f(u)}{8\pi GL^5 u} \left(h_t^{\prime\prime\alpha} + h_t^{\prime\alpha} \left[\frac{f'(u)}{f(u)} - \frac{4}{u} \right] + h_t^\alpha \left[\frac{6}{u^2} - \frac{3f'(u)}{uf(u)} \right] \right) \quad (61)$$

In order to handle the ϕ_i'' terms, we evaluate the above expressions on-shell, yielding

$$(\delta J^\alpha)' = -\frac{r_H^2}{8\pi GL^3} \bar{\kappa} ik\epsilon_{\alpha\beta} a_\beta \quad (62)$$

$$(\delta t_t^\alpha)' = \frac{r_H^4 f(u)}{8\pi GL^5 u} \left(h_t^{\prime\alpha} \left[\frac{f'(u)}{f(u)} - \frac{3}{u} \right] + h_t^\alpha \left[\frac{6}{u^2} - \frac{3f'(u)}{uf(u)} \right] + \frac{3au}{\mu} a'_\alpha \right) \quad (63)$$

²At zero frequency the fields h_y^α decouple from the system and thus will not be considered (see [8]).

Now, observe that, since $h_t'^\alpha = \frac{8\pi GL^5 u}{r_H^4 f(u)} \delta t_t^\alpha + \frac{3}{u} h_t^\alpha$ and $a'_\alpha = \left(\frac{8\pi GL^3}{r_H^2} \delta J^\alpha + \mu h_t^\alpha \right) \frac{1}{f(u)}$, (63) turns into

$$(\delta t_t^\alpha)' = \frac{3r_H^4 f(u)}{8\pi GL^5 u} \left[\frac{au}{f(u)} - \frac{1}{u^2} \right] h_t^\alpha + \left[\frac{f'(u)}{f(u)} - \frac{3}{u} \right] \delta t_t^\alpha + \frac{3r_H^2 a}{L^2 \mu} \delta J^\alpha \quad (64)$$

Plugging the constitutive relations ($\epsilon^{xz} \equiv 1$)

$$\delta J_{const}^\alpha = G^{xx} \delta^{\alpha\beta} a_\beta + G^{xz} \epsilon^{\alpha\beta} a_\beta + P^{xt} \delta^{\alpha\beta} h_\beta^t + P^{zt} \epsilon^{\alpha\beta} h_\beta^t \quad (65)$$

$$\delta t_{const}^\alpha = G_\epsilon^{xx} \delta^{\alpha\beta} a_\beta + G_\epsilon^{xz} \epsilon^{\alpha\beta} a_\beta + P_\epsilon^{xt} \delta^{\alpha\beta} h_\beta^t + P_\epsilon^{zt} \epsilon^{\alpha\beta} h_\beta^t \quad (66)$$

into (64), the remaining equation for $(\delta t_t^\alpha)'$ involves only ϕ_i and G_j^i .

On the other hand, we can take the u-derivative of (65)-(66) explicitly and then make use of (58)-(59) to end up having an equation in terms of ϕ_i , G_j^i and G_j^i .

Finally, imposing $(\delta J_{const}^\alpha)' = (\delta J^\alpha)'$ and $(\delta t_{const}^\alpha)' = (\delta t_t^\alpha)'$ and assuming that the perturbations ϕ_i are independent from each other, we find

$$G'^{xx} + \frac{8\pi GL^3}{f(u)r_H^2} ((G^{xx})^2 - (G^{xz})^2) - \frac{8\pi GL^5 u}{r_H^4 f^2(u)} (P^{xt} G_\epsilon^{xx} - P^{zt} G_\epsilon^{xz}) = 0 \quad (67)$$

$$G'^{xz} + \frac{16\pi GL^3}{f(u)r_H^2} G^{xx} G^{xz} - \frac{8\pi GL^5 u}{r_H^4 f^2(u)} (P^{xt} G_\epsilon^{xz} + P^{zt} G_\epsilon^{xx}) = -\frac{r_H^2}{8\pi GL^3} \bar{\kappa} i k \quad (68)$$

$$P'^{xt} + G^{xx} \left(-\mu + \frac{8\pi GL^3}{f(u)r_H^2} P^{xt} \right) - \left(\frac{8\pi GL^3}{f(u)r_H^2} G^{xz} - \frac{8\pi GL^5 u}{r_H^4 f^2(u)} P_\epsilon^{zt} \right) P^{zt} + \\ + P^{xt} \left(-\frac{8\pi GL^5 u}{r_H^4 f^2(u)} P_\epsilon^{xt} - \frac{f'(u)}{f(u)} + \frac{3}{u} \right) = 0 \quad (69)$$

$$P'^{zt} + G^{xz} \left(-\mu + \frac{8\pi GL^3}{f(u)r_H^2} P^{xt} \right) - \frac{8\pi GL^5 u}{f^2(u)r_H^4} P^{xt} P_\epsilon^{zt} + \\ + P^{zt} \left(-\frac{8\pi GL^5 u}{f^2(u)r_H^4} P_\epsilon^{xt} + \frac{3}{u} - \frac{f'(u)}{f(u)} + \frac{8\pi GL^3}{f(u)r_H^2} G^{xx} \right) = 0 \quad (70)$$

$$G'_\epsilon{}^{xx} + \frac{8\pi GL^3}{f(u)r_H^2} (G_\epsilon^{xx}G^{xx} - G_\epsilon^{xz}G^{xz}) - \frac{8\pi GL^5 u}{r_H^4 f^2(u)} P_\epsilon^{xt} G_\epsilon^{xx} + \frac{8\pi GL^5 u}{r_H^4 f^2(u)} P_\epsilon^{zt} G_\epsilon^{xz} =$$

$$= -G_\epsilon^{xx} \left(\frac{3}{u} - \frac{f'(u)}{f(u)} \right) + \mu G^{xx} \quad (71)$$

$$G'_\epsilon{}^{xz} + \frac{8\pi GL^3}{f(u)r_H^2} (G_\epsilon^{xx}G^{xz} + G_\epsilon^{xz}G^{xx}) - \frac{8\pi GL^5 u}{r_H^4 f^2(u)} (P_\epsilon^{xt} G_\epsilon^{xz} + P_\epsilon^{zt} G_\epsilon^{xx}) =$$

$$= -G_\epsilon^{xz} \left(\frac{3}{u} - \frac{f'(u)}{f(u)} \right) + \mu G^{xz} \quad (72)$$

$$P'_\epsilon{}^{xt} + G_\epsilon^{xx} \left(-\mu + \frac{8\pi GL^3}{f(u)r_H^2} P^{xt} \right) - \frac{8\pi GL^3}{f(u)r_H^2} G_\epsilon^{xz} P^{zt} + P_\epsilon^{xt} \left(-\frac{8\pi GL^5 u}{r_H^4 f^2(u)} P_\epsilon^{xt} + \frac{3}{u} - \frac{f'(u)}{f(u)} \right) +$$

$$+ \frac{8\pi GL^5 u}{r_H^4 f^2(u)} (P_\epsilon^{zt})^2 = -P_\epsilon^{xt} \left(\frac{3}{u} - \frac{f'(u)}{f(u)} \right) + \mu P^{xt} - \frac{3r_H^4}{8\pi GL^5 u} f(u) \left(au - \frac{f(u)}{u^2} \right) \quad (73)$$

$$P'_\epsilon{}^{zt} + G_\epsilon^{xz} \left(\frac{8\pi GL^3}{f(u)r_H^2} P^{xt} - \mu \right) + \frac{8\pi GL^3}{f(u)r_H^2} G_\epsilon^{xx} P^{zt} - \frac{8\pi GL^5 u}{f^2(u)r_H^4} P_\epsilon^{xt} P_\epsilon^{zt} +$$

$$+ P_\epsilon^{zt} \left(-\frac{8\pi GL^5 u}{f^2(u)r_H^4} P_\epsilon^{xt} + \frac{3}{u} - \frac{f'(u)}{f(u)} \right) = -P_\epsilon^{zt} \left(\frac{3}{u} - \frac{f'(u)}{f(u)} \right) + \mu P^{zt} \quad (74)$$

By directly studying the structure of the solutions to (56)-(57), it can be realized that $G^{xx} = P^{xt} = G_\epsilon^{xx} = 0$ for $\omega = 0$ and to first order in k . Furthermore, all the anomalous correlators are of order k or higher. A more detailed study of (67)-(74) is left for section 3.3.1.

3.3 Flow of the transport coefficients as two point functions

As suggested in Section 2, we could have determined the flow by simply considering the system to be restricted to live between the horizon and a cutoff surface placed at Λ . It is hence expected that the transport coefficients at the boundary can be computed by finding the corresponding 2-point functions. The boundary value of the perturbations, whose bulk-to-boundary propagator is normalized at the cutoff, work as the sources for the different operators of the dual theory.

Henceforth, the perturbations will be rearranged in a vector $\Phi(u, x^\mu)$. It is more convenient to use the Fourier transformed quantity

$$\Phi(u, x^\mu) = \int \frac{d^d k}{(2\pi)^d} \Phi_k^I(u) e^{-i\omega t + i\vec{k}\vec{x}} \quad (75)$$

The explicit expression for $\Phi_k(u)$ is

$$\Phi_k^\top(u) = (B_x(u), h_t^x(u), B_z(u), h_t^z(u)) \quad (76)$$

being $B_\alpha = a_\alpha/\mu$. To proceed, one can follow [22] and assume the general form of a boundary action

$$\delta S^{(2)} = \int_{r=\Lambda} \frac{d^d k}{(2\pi)^d} [\Phi_{-k}^I \mathcal{A}_{IJ} \Phi_k'^J + \Phi_{-k}^I \mathcal{B}_{IJ} \Phi_k^J] \quad (77)$$

In order to get the solution of the system (56)-(57) to first order in momentum we expand the fields in the (dimensionless) quantity $p = \frac{k}{4\pi T}$. Hence

$$h_t^\alpha(u) = h_t^{(0),\alpha}(u) + p h_t^{(1),\alpha}(u) \quad (78)$$

$$B_\alpha(u) = B_\alpha^{(0)}(u) + p B_\alpha^{(1)}(u) \quad (79)$$

The system can be solved perturbatively. To calculate the retarded correlators at $r = \Lambda$ (or, equivalently, at $u = u_c \equiv r_H^2/\Lambda^2$) we only need to solve the equations for the perturbations with infalling boundary conditions, on the one hand, and boundary conditions $\Phi_k^I(u_c) = \phi_k^I$ on the other [9]. This procedure should give us the desired Green's functions, after taking the variation of (77) with respect to the fields at $u = u_c$ (which act as sources for their corresponding operators). Recall that, as explained in Section 2, the bulk-to-boundary propagator must be normalized at $r = \Lambda$, that is, if we have

$$\Phi_k^I(u) = F_J^I(k, u) \phi_k^J \quad (80)$$

then $F_J^I(k, u_c) = 1$. Notice that the relation between the boundary value at $u = u_c$ and that at $u = 0$ is simply $\phi_k^{I(u_c)} = F_J^I(k, u_c) \phi_k^{J(0)}$, so that the solution is preserved by these manipulations, as pointed out by [13] and [16]. The retarded two-point functions, from which we are able to read directly the transport coefficients, then have the form

$$G_{IJ}(k, u_c) = -2 \lim_{u \rightarrow u_c} \left(\mathcal{A}_{IM} (F_J^M(k, u))' + \mathcal{B}_{IJ} \right) \quad (81)$$

Where the \mathcal{A}_{IJ} and \mathcal{B}_{IJ} matrices are [9]

$$\mathcal{A} = \frac{r_H^4}{16\pi GL^5} \text{Diag} \left(-3af(u), \frac{1}{u}, -3af(u), \frac{1}{u} \right) \quad (82)$$

$$B_{AdS+\partial} = \frac{r_H^4}{16\pi GL^5} \begin{bmatrix} 0 & 3a & 0 & 0 \\ 0 & -\frac{3}{u^2} & 0 & 0 \\ 0 & 0 & 0 & 3a \\ 0 & 0 & 0 & -\frac{3}{u^2} \end{bmatrix} \quad (83)$$

Using again the the effective chemical potential

$$\mu(\Lambda) = \mu \left(1 - \frac{r_H^2}{\Lambda^2} \right), \quad (84)$$

the result for the anomalous correlators is

$$\langle \delta J^x \delta J^z \rangle = \frac{i\mu\kappa k}{2\pi G} \left(1 - \frac{r_H^2}{\Lambda^2} \right) = \frac{-ik\mu(\Lambda)}{4\pi^2} \quad (85)$$

$$\langle \delta J^x \delta t_t^z \rangle = \langle \delta t_t^x \delta J^z \rangle = -\frac{i\kappa\mu^2 k}{4\pi G} \left(1 - \frac{r_H^2}{\Lambda^2} \right)^2 = \frac{ik\mu(\Lambda)^2}{8\pi^2} \quad (86)$$

$$\langle \delta t_t^x \delta t_t^z \rangle = \frac{i\kappa\mu^3 k}{6\pi G} \left(1 - \frac{r_H^2}{\Lambda^2} \right)^3 = \frac{-ik\mu(\Lambda)^3}{12\pi^2} \quad (87)$$

Since $\lim_{\Lambda \rightarrow \infty} \mu(\Lambda) = \mu$, these correlators coincide essentially with the ones derived in [9]³.

3.3.1 Compatibility with the flow equations

The system of first order differential equations (67)-(74) must be compatible with the result (85)-(87) encountered in the previous section. In order to check that it is so, the dissipative correlators play an important role. In the case $\omega = 0$ and to $\mathcal{O}(k)$, they read⁴

$$G^{xx} = P^{xt} = G_\epsilon^{xx} = 0 \quad (88)$$

$$P_\epsilon^{xt} = -\frac{r_H^4}{8\pi GL^5 u} f^2(u) \left(\frac{f'(u)}{f(u)} - \frac{3}{u} \right) \quad (89)$$

This solution implies that G^{xx} and $P^{xt} = G_\epsilon^{xx}$ are of order ω or higher, whereas P_ϵ^{xt} contains a part which is of order $\mathcal{O}(k^0, \omega^0)$ (contact term). The remaining system, after substituting (88), (89) and assuming that all the anomalous correlators are at least of $\mathcal{O}(k)$, turns out to be (up to order k)

³The minus sign found in (86) with respect to the result of [9] is due to the fact that in this reference the correlator that is studied is $\langle \delta J^a \delta t_b^t \rangle$, that differs from $\langle \delta J^a \delta t_t^b \rangle$ by a factor of (b represents a spatial index) $g_{tt}g^{bb} = -f(u) \rightarrow -1$ at infinity.

⁴The limit $P_\epsilon^{xt}(u=0)$ is not well defined because we have not included the corresponding counterterms in (58),(59). The reason is that they do not affect the anomalous correlators.

$$G'^{xx} = 0 \quad (90)$$

$$G'^{xz} = -\frac{r_H^2}{8\pi GL^3} \bar{\kappa} i k \quad (91)$$

$$P'^{xt} = 0 \quad (92)$$

$$P'^{zt} - \mu G^{xz} = 0 \quad (93)$$

$$G_\epsilon'^{xx} = 0 \quad (94)$$

$$G_\epsilon'^{xz} = \mu G^{xz} \quad (95)$$

$$P_\epsilon'^{xt} = -P_\epsilon^{xt} \left(\frac{3}{u} - \frac{f'(u)}{f(u)} \right) - \frac{3r_H^4}{8\pi GL^5 u} f(u) \left(au - \frac{f(u)}{u^2} \right) \quad (96)$$

$$P_\epsilon'^{zt} - \mu G_\epsilon'^{xz} = \mu P^{zt} \quad (97)$$

Equation (96) is in agreement with (89). In the end, the 2-point functions associated with dissipative transport coefficients decouple completely. Regarding the anomalous correlators, the above system of equations can be integrated easily, leading to

$$G^{xz} = \frac{r_H^2}{8\pi GL^3} \bar{\kappa} i k (1 - u_c) \quad (98)$$

$$P^{zt} = G_\epsilon^{xz} = -\mu \frac{r_H^2}{16\pi GL^3} \bar{\kappa} i k (1 - u_c)^2 \quad (99)$$

$$P_\epsilon^{xt} = \mu^2 \frac{r_H^2}{24\pi GL^3} \bar{\kappa} i k (1 - u_c)^3 \quad (100)$$

which is the same as (85)-(87). The role played by the Chern-Simons term in (52) is crucial to ensure that G^{xz} presents a flow, for in its absence all the anomalous 2-point functions identically vanish.

4 Gravitational Anomaly

The study of the effect of the Gravitational Anomaly on the definition of the holographic operators is a non-trivial task, for the term $A \wedge R \wedge R$ has not a well defined Dirichlet problem. This makes, strictly speaking, not possible to define generic operators. In [8], the problem was circumvented by arguing that any possible contribution vanishes asymptotically. However, now we are interested in the value of the transport coefficients at finite cutoff Λ , and therefore it is necessary to face this issue.

4.1 The Model

The four dimensional axial gravitational anomaly is induced holographically by a Chern-Simons term of the form [8]

$$S_{ACS} = \frac{\lambda}{16\pi G} \int d^5x \sqrt{-g} \epsilon^{MNPQR} A_M R_{BNP}^A R_{AQR}^B \quad (101)$$

This action contributes to the boundary axial current as expected for a mixed anomaly. The complete action reads

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} \left[R + 2\Lambda_c - \frac{1}{4} F_{MN} F^{MN} \right] + S_{ACS} + S_{AEM} + S_\partial + S_{CSK} \quad (102)$$

Where

$$S_{AEM} = \frac{\kappa}{48\pi G} \int d^5x \sqrt{-g} \epsilon^{MNPQR} A_M F_{NP} F_{QR}, \quad (103)$$

$$S_\partial = -\frac{1}{8\pi G} \int_\partial \sqrt{-h} K, \quad (104)$$

$$S_{CSK} = -\frac{\lambda}{2\pi G} \int_{\partial\mathcal{M}} d^4x \sqrt{-h} n_M \epsilon^{MNPQR} A_N K_{PL} D_Q K_R^L. \quad (105)$$

Adding S_{CSK} ensures that the anomalous Ward identity for gauge transformations depends only on the intrinsic curvature tensor on the boundary at $r = \Lambda$ [8].

Indeed, the covariant current turns out to be

$$16\pi G J^A = n_B \left[F^{AB} - 8\epsilon^{BACDE} \lambda K_{CF} D_D K_F^E \right]_{r=\Lambda} \quad (106)$$

with a purely four dimensional divergence that on shell evaluates to

$$D_\mu J^\mu = -\frac{1}{16\pi G} \epsilon^{opqr} \left[\frac{\kappa}{3} F_{op} F_{qr} + \lambda R_{(4)bop}^a R_{(4)aqr}^b \right]_{r=\Lambda} \quad (107)$$

where $\epsilon^{opqr} \equiv \epsilon^{nopr}$ is the four dimensional epsilon tensor.

The bulk equations of motion are

$$G_{MN} - \Lambda_c g_{MN} = \frac{1}{2} F_{ML} F_N{}^L - \frac{1}{8} F^2 g_{MN} + 2\lambda \epsilon_{LPQR(M} \nabla_B (F^{PL} R_{N)}^B{}^{QR}), \quad (108)$$

$$\nabla_N F^{NM} = -\epsilon^{MNPQR} (\kappa F_{NP} F_{QR} + \lambda R_{BNP}^A R_{AQR}^B), \quad (109)$$

4.2 Contribution of the Gravitational Anomaly

If we vary S_{ACS} , we are left with a term which spoils the variational problem

$$\frac{\lambda}{2\pi G} \int_{\partial} \sqrt{-h} \epsilon^{mlqr} A_m D_r K_q^v \delta K_{lv} \quad (110)$$

If we looked for a suitable counterterm to render the Dirichlet problem well-posed, we would end up finding $S_{CS\mathcal{K}}$. Indeed, this boundary contribution was firstly conceived as an analogue to the Gibbons-Hawking-York term. However, after varying $S_{ACS} + S_{CS\mathcal{K}}$ one realizes that the result

$$- \frac{\lambda}{2\pi G} \int_{\partial} \sqrt{-h} \epsilon^{mlqr} D_r A_m \delta K_q^v K_{lv} \quad (111)$$

is still problematic. Even worse, (111) can not be canceled easily, for, for instance, the ansatz

$$\frac{\lambda}{2\pi G} \int_{\partial} \sqrt{-h} \epsilon^{mlqr} D_r A_m K_q^v K_{lv} \quad (112)$$

is automatically zero. Thus in principle, there is not a straightforward way of having a well defined variational problem for this system.

On the other hand, as aforementioned, we need $S_{CS\mathcal{K}}$ to have a four dimensional anomalous Ward identity at the boundary, so we will keep it. A hypothetical generic counterterm (if it exists) capable of solving all the problems, would probably ruin (107) and therefore, by physical means, should not be considered.

Even though the variational problem is not well-posed, we will still be able to derive the equations of motion by means of the analogue of the Euler-Lagrange equations for higher-derivative theories. The difficulty therefore reduces to the question *How to treat (111) holographically?* Note that in [22] it was implicitly assumed that the Dirichlet problem is correctly defined, so we should go a little bit further in this case.

Specializing for the shear sector, which is the one that interests us, and at second order in perturbations, (111) reads

$$- \frac{\lambda}{2\pi G} \int_{\partial} \sqrt{-h} \epsilon^{mlqr} D_r \delta A_m \delta K_q^v K_{lv} \quad (113)$$

Other possible terms would vanish in the background (53). The strategy would be the following: Since (113) does not affect two point functions involving only energy-momentum tensors or only currents, we know how to compute the correlators $\langle T_t^x T_t^z \rangle$ and $\langle J^x J^z \rangle$. (113) only plays a role when calculating $\langle T_t^x J^z \rangle$, $\langle J^x T_t^z \rangle$, and hence those are the ones for which the discussion of [22] does not apply.

Following the method detailed in Section 3, it turns out that, taking only into account the gravitational anomaly

$$\langle J^x J^z \rangle = 0 \quad (114)$$

$$\langle T_t^x T_t^z \rangle = -ik \frac{\mu(1-u_c)T^2}{12} \quad (115)$$

(note that we have directly substituted the value of λ for a single left-handed fermion $\lambda/G = -\frac{1}{48\pi}$). The above results point again towards an effective $\mu(1-u_c)$. Therefore, by physical grounds, we expect the appearance of an effective temperature also. Note that the flows of the effective quantities must be consistent in the sense that they must be the same, no matter what correlator we are focusing on. Equations (114)-(115) hint at the existence of an effective temperature for the system; this temperature does not flow with the cutoff scale, being always identical to the Hawking temperature. This conclusion is in agreement with the asymptotic values of [8].

So we resolve that (111) must be treated in such a way that $\langle T_t^x J^z \rangle$, $\langle J^x T_t^z \rangle$, at finite cutoff, are consistent with a non-flowing temperature.

It turns out that the method to achieve it is precisely the one that one would anticipate by general considerations: Taking advantage of the fact that the equations of motion for the shear sector

$$\begin{aligned} 0 = & h_t^{\alpha''}(u) - \frac{h_t^{\alpha'}(u)}{u} - 3auB'_\alpha(u) + i\bar{\lambda}k\epsilon_{\alpha\beta} \left[(24au^3 - 6(1-f(u))) \frac{B_\beta(u)}{u} \right. \\ & \left. + (9au^3 - 6(1-f(u)))B'_\beta(u) + 2u(uh_t^{\beta'}(u))' \right], \end{aligned} \quad (116)$$

$$\begin{aligned} 0 = & B''_\alpha(u) + \frac{f'(u)}{f(u)}B'_\alpha(u) - \frac{h_t^{\alpha'}(u)}{f(u)} \\ & + ik\epsilon_{\alpha\beta} \left(\frac{3}{uf(u)}\bar{\lambda} \left(\frac{2}{a}(f(u)-1) + 3u^3 \right) h_t^{\beta'}(u) + \bar{\kappa} \frac{B_\beta(u)}{f(u)} \right), \end{aligned} \quad (117)$$

happen to be of second order in derivatives (where $\bar{\lambda} = \frac{4\mu\lambda L}{r_H^2}$), we can solve completely the evolution as we did in Section 3.3 (imposing in-falling B.C. at the Horizon and Dirichlet B.C. at the boundary). Once the solutions are known (see the appendix), (111) will in general give a well determined surface contribution (when evaluated on-shell) that must be taken into account to calculate $\langle T_t^x J^z \rangle$, $\langle J^x T_t^z \rangle$. The result so obtained presents no flow in the temperature part.

To be more concise, the boundary term (111) to be considered has the following form

$$- \frac{ik\lambda r_H^2 \epsilon_{\alpha\beta}}{2\pi GL^4} \int_{\partial} u f'(u) a_\beta(k) h_t^{\prime\alpha}(-k) \quad (118)$$

whose contribution, up to first order in k , is summarized

$$- \frac{ik\lambda r_H^2 \epsilon_{\alpha\beta}}{2\pi GL^4} \int_{\partial} u \frac{f'^2(u)}{f(u_c)} a_{\beta}^{(0)}(k) \tilde{H}_{\beta}^{(0)}(-k) \quad (119)$$

(Notice the factor $\sim \frac{1}{f(\Lambda)}$ introduced to normalize the perturbation (see the appendix)). So the effect of (111) on the Green's functions can be reformulated as a modification, prescribed by (119), of the \mathcal{B}_{IJ} matrix.

Even though (119) only affects the correlator $\langle T_t^{\alpha} J^{\beta} \rangle$, $S_{\mathcal{ACS}} + S_{\mathcal{CSK}}$ induces automatically a non-vanishing value for the components $\mathcal{A}_{14} = \mathcal{A}_{32}^*$ of the matrix \mathcal{A} . These contributions, which are perfectly treatable within the framework of [22], give rise to a correction of $\langle J^{\alpha} T_t^{\beta} \rangle$ which is precisely of the same form of the one implemented by (119). As will be mentioned below, this turns out to be sufficient for the consistency condition (127) to hold.

The final form of the matrices \mathcal{A}_{IJ} and \mathcal{B}_{IJ} after implementing the shift driven by the Gravitational Anomaly is given by

$$\mathcal{A} = \frac{r_H^4}{16\pi GL^5} \begin{bmatrix} -3af(u) & 0 & 0 & -\frac{4i\lambda k L}{r_H^2} u f'(u) \\ 0 & \frac{1}{u} & 0 & \frac{i8\lambda k L \mu}{r_H^2} u \\ 0 & \frac{4i\lambda k L}{r_H^2} u f'(u) & -3af(u) & 0 \\ 0 & -\frac{i8\lambda k L \mu}{r_H^2} u & 0 & \frac{1}{u} \end{bmatrix} \quad (120)$$

$$B_{AdS+\partial} = \frac{r_H^4}{16\pi GL^5} \begin{bmatrix} 0 & 3a & 0 & 0 \\ 0 & -\frac{3}{u^2} & 4i\lambda k L \frac{9au^3 - 6(1-f(u))}{ur_H^2} & 0 \\ 0 & 0 & 0 & 3a \\ -4i\lambda k L \frac{9au^3 - 6(1-f(u))}{ur_H^2} & 0 & 0 & -\frac{3}{u^2} \end{bmatrix} \quad (121)$$

$$B_{\partial CS} = \frac{r_H^4}{16\pi GL^5} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{4i\lambda k L}{r_H^2} u \frac{f'(u)^2}{f(u_c)} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{4i\lambda k L}{r_H^2} u \frac{f'(u)^2}{f(u_c)} & 0 & 0 & 0 \end{bmatrix} \quad (122)$$

The resulting anomalous 2-point functions are

$$\langle J^x J^z \rangle = \frac{ik\kappa\mu(1-u_c)}{2G\pi} = -ik\frac{\mu(\Lambda)}{4\pi^2} \quad (123)$$

$$\langle J^x T_t^z \rangle = -\frac{ik\kappa(1-u_c)^2\mu^2}{4G\pi} - \frac{ik\lambda(-2+a)^2 r_H^2}{2GL^4\pi} = ik\left(\frac{\mu^2(1-u_c)^2}{8\pi^2} + \frac{T^2}{24}\right) \quad (124)$$

$$\langle T_t^x J^z \rangle = -\frac{ik\kappa(1-u_c)^2\mu^2}{4G\pi} - \frac{ik\lambda(-2+a)^2 r_H^2}{2GL^4\pi} = ik\left(\frac{\mu^2(1-u_c)^2}{8\pi^2} + \frac{T^2}{24}\right) \quad (125)$$

$$\langle T_t^x T_t^z \rangle = \frac{ik\kappa(1-u_c)^3\mu^3}{6G\pi} + (1-u_c)\mu\frac{ik\lambda(-2+a)^2 r_H^2}{GL^4\pi} = -ik\left(\frac{\mu^3(1-u_c)^3}{12\pi^2} + \frac{\mu(\Lambda)T^2}{12}\right) \quad (126)$$

Observe that it is straightforward to verify that equations (123)-(126) are compatible with the asymptotic value computed in [8]. Notice also that the temperature part remains constant as we move the boundary. The flow of the different correlators is consistent with respect to each other and the hypothesis of an effective chemical potential $\mu(\Lambda) = \mu\left(1 - \frac{r_H^2}{\Lambda^2}\right) \equiv \mu(1-u_c)$ is reinforced by the results extracted from the terms proportional to λ .

5 Discussion and Conclusion

We have studied the holographic cutoff flow of the anomalous transport coefficients. This has been done by defining a bottom up model that implements both the axial and the mixed axial-gravitational anomalies. The flow has been studied by analyzing the dependence of the anomalous Green's functions on the radial position, Λ , of the boundary. We have presented several prescriptions to compute such flow and finally obtained it by adapting the method implemented in [8], [9] for the case $\Lambda \rightarrow \infty$.

It is a remarkable fact that the chiral magnetic conductivity suffers from a flow even in the non-backreacted case. In fact, this could have been anticipated by noticing that regularity at the horizon imposes that in the deep IR the constitutive relations are only compatible with an electric conductivity ([13]), so that if a system exhibits a chiral magnetic conductivity in the UV it must be due to a non-trivial flow.

When considering the gravitational anomaly, a Dirichlet boundary condition is not enough anymore to define the variational problem properly. A generic definition of suitable operators, if any, therefore requires further discussion in this case. In this paper we have simply focused on computing 2-point functions, without discussing general definitions of the corresponding operators. The term which spoils the variational principle has been dealt with by considering its effect on the on-shell action. This procedure, which can be seen to be the most natural one by using physical arguments, yields 2-point functions that are consistent and whose flows do not get in contradiction with the result found in the absence of gravitational anomaly. Moreover, in the spirit of [22], that the matrix of correlators \mathcal{G}_{IJ} obeys

$$\frac{d}{du} (\mathcal{G} - \mathcal{G}^\dagger) = 0, \quad (127)$$

represents a non-trivial consistency check.

The result (123)-(126) shows that the temperature remains constant (Hawking temperature) whereas the chemical potential presents a flow that is easily interpretable in terms of the energy necessary to bring a unit of charge from the horizon to the boundary. Observe, however, that all the correlators are written for a metric with $g_{tt} \sim -r^2 f(r)$, and hence there is an implicit redshift factor between observers living in one hypersurface placed at $r = \Lambda$ and another one at $r = \Lambda'$.

From the point of view of the boundary theory, these outcomes indicate that the pure gauge Chern-Simons term does not affect the boundary operators but influences the anomalous correlators through the flow equations, forcing them to have a non-vanishing value at the boundary, whereas the gravitational-gauge Chern-Simons term happens not to have any impact by means of the evolution equations, but to induce new covariant contributions, that are first order in k , to the operators, so that the constant T^2 part is present at any value of the r -coordinate.

Appendix:

Solutions at zero frequency and normalized at finite cut-off u_c

Case $\lambda = 0$

$$\begin{aligned} B^\alpha(u) = & \bar{B}^\alpha + \bar{H}^\alpha(u - u_c) - \frac{i\bar{k}k\epsilon_{\alpha\beta}}{2(1+4a)^2(-1+u_c(-1+au_c))} \times \\ & \times ((1+4a)(u - u_c)(\bar{H}^\beta + \bar{H}^\beta u_c + a(3\bar{B}^\beta(2+u_c) + \bar{H}^\beta(4 - u_c(2+3u_c)))) + \\ & + 2\sqrt{1+4a}(-2+a(-2+3u))(\bar{B}^\beta - \bar{H}^\beta u_c)(-1+u_c(-1+au_c))(\text{ArcTanh} \left[\frac{-1+2au}{\sqrt{1+4a}} \right] + \\ & + \text{ArcTanh} \left[\frac{1-2au_c}{\sqrt{1+4a}} \right])) \end{aligned} \quad (128)$$

$$\begin{aligned}
H_t^\alpha(u) = & -\frac{1}{2(-1-4a)^{3/2}(-1+u_c)(-1+u_c(-1+au_c))^2}(-1+u) \times \\
& \times (-2(-1-4a)^{3/2}\bar{H}^\alpha(-1+u(-1+au))(-1+u_c(-1+au_c)) + \\
& + k\bar{\kappa}\epsilon_{\alpha\beta}(-i(\sqrt{-1-4a}-i\sqrt{1+4a})\bar{H}^\beta(1+u)(1+u_c) + \\
& + a^2 3\bar{B}^\beta(2i\sqrt{-1-4a}u_c^2 + i\sqrt{-1-4a}uu_c^2 + \sqrt{1+4a}u^2(2+u_c)) + \\
& + a^2\bar{H}^\beta(2i\sqrt{-1-4a}(2-3u_c)u_c^2 + i\sqrt{-1-4a}u(4-3u_c)u_c^2 + \sqrt{1+4a}u^2(4-u_c(2+3u_c))) - \\
& - 3ia\bar{B}^\beta(2\sqrt{-1-4a}-2i\sqrt{1+4a}+2\sqrt{-1-4a}u_c-i\sqrt{1+4a}u_c) - \\
& - 3iau\bar{B}^\beta(\sqrt{-1-4a}-2i\sqrt{1+4a}+\sqrt{-1-4a}u_c-i\sqrt{1+4a}u_c) + \\
& + a\bar{H}^\beta(-4i\sqrt{-1-4a}-4\sqrt{1+4a}+\sqrt{1+4a}u^2(1+u_c)) + \\
& + a\bar{H}^\beta u_c(2i\sqrt{-1-4a}+2\sqrt{1+4a}+7i\sqrt{-1-4a}u_c+3\sqrt{1+4a}u_c) + \\
& + ua\bar{H}^\beta(-4i\sqrt{-1-4a}-4\sqrt{1+4a}) + \\
& + ua\bar{H}^\beta u_c(-i\sqrt{-1-4a}+2\sqrt{1+4a}+4i\sqrt{-1-4a}u_c+3\sqrt{1+4a}u_c)) + \\
& + 6iak(-1+u(-1+au))\bar{\kappa}\epsilon_{\alpha\beta}(\bar{B}^\beta-\bar{H}^\beta u_c)(-1+u_c(-1+au_c))\text{ArcTan}\left[\frac{-1+2au}{\sqrt{-1-4a}}\right] + \\
& + 6ak(-1+u(-1+au))\bar{\kappa}\epsilon_{\alpha\beta}(\bar{B}^\beta-\bar{H}^\beta u_c)(-1+u_c(-1+au_c))\text{ArcTanh}\left[\frac{1-2au_c}{\sqrt{1+4a}}\right]
\end{aligned}
\tag{129}$$

Case $\kappa = 0$

$$\begin{aligned}
B^\alpha(u) = & \bar{B}^\alpha + \bar{H}^\alpha(u - u_c) + \\
& + \frac{1}{6(2-a)a^3}(-2+a)k\bar{u}_c\epsilon_{\alpha\beta}\left(\frac{2i(-2+a(-2+3u))\text{ArcTanh}\left[\frac{1-2au}{\sqrt{1+4a}}\right]}{(1+4a)^{3/2}}\right) \times \\
& \times (4\bar{H}^\beta + a(3(1+a(7+2a(7+a))))\bar{B}^\beta + 4(8+a(2+a)(9+2a))\bar{H}^\beta - \\
& - 3(1+a(7+2a(7+a)))\bar{H}^\beta u_c)) + \\
& + \frac{a}{b}(-2a\sqrt{1+4a}(u-u_c)(6a(\bar{B}^\beta - \bar{H}^\beta(-8+u)) + 6a(\bar{B}^\beta - \bar{H}^\beta(-8+u))u_c - 8a\bar{H}^\beta u_c^2 - \\
& - 3a^2(3\bar{B}^\beta(-4+u)(1+u_c) + \bar{H}^\beta(u(10+3u) + u(10+3u)u_c - 2(-8+u)u_c^2 - 4(7+6u_c))) + \\
& + 8\bar{H}^\beta(1+u_c)) + \\
& + a^4(\bar{H}^\beta(12+u_c(18+(-59+12u(2+3u))u_c)) + 3\bar{B}^\beta(2+u_c(5+12(-1+u-u_c)u_c))) + \\
& + a^3 9\bar{B}^\beta(4+u(-4+(-4+u_c)u_c) - u_c(-5+u_c+u_c^2)) + \\
& + a^3 \bar{H}^\beta(29+(23-72u_c)u_c + 9u^2(-4+(-4+u_c)u_c) + 6u(-4+u_c(-4+5u_c))) + \\
& + (-2+a(-2+3u))(-1+u_c(-1+au_c))(2(4\bar{H}^\beta + a(3(1+a(7+2a(7+a))))\bar{B}^\beta + \\
& + 4(8+a(2+a)(9+2a))\bar{H}^\beta - 3(1+a(7+2a(7+a)))\bar{H}^\beta u_c))\text{ArcTanh}\left[\frac{1-2au_c}{\sqrt{1+4a}}\right] + \\
& + (1+a)(1+4a)^{3/2}(-4\bar{H}^\beta + a(-3\bar{B}^\beta - 4\bar{H}^\beta + 3\bar{H}^\beta u_c)) \times \\
& \times (\text{Log}[-1+u(-1+au)] - \text{Log}[-1+u_c(-1+au_c)]))
\end{aligned} \tag{130}$$

where $\frac{a}{b} \equiv \frac{1}{(-1-4a)^{3/2}(1+u_c-au_c^2)}$

$$\begin{aligned}
H_t^\alpha(u) = & \bar{H}^\alpha \frac{(-1+u)(-1+u(-1+au))}{(-1+u_c)(-1+u_c(-1+au_c))} + \frac{(1-u)\epsilon_{\alpha\beta}}{2(-1-4a)^{3/2}a^2} k(1+u-au^2) \bar{u}_c \times \\
& \times \left(-\frac{1}{(-1+u(-1+au))(-1+u_c(-1+au_c))} 2a\sqrt{1+4a}(u-u_c) \times \right. \\
& \times (4\bar{H}^\beta(1+u)(1+u_c) + a\bar{B}^\beta(3+5u+5(1+u)u_c) + \\
& + a\bar{H}^\beta(18+u(25+u)+22u_c+u(25+u)u_c-4(1+u)u_c^2) - \\
& - 3a^2\bar{B}^\beta(-5+2u^2(1+u_c)+u_c(-7+2u_c)+u(-7+2(-3+u_c)u_c)) + \\
& + \bar{H}^\beta a^2(18+24u_c-22u_c^2-6u^3(1+u_c)+u(39+5(7-5u_c)u_c)-u^2(1+u_c+u_c^2)) + \\
& + 3a^3\bar{B}^\beta(1-8u_c^2-uu_c(3+8u_c)+2u^2(-4+(-4+u_c)u_c)) + \\
& + \bar{H}^\beta a^3(4-4u_c(2+5u_c)+u^2(-20+(-20+u_c)u_c)+6u^3(-4+(-4+u_c)u_c)) - \\
& - \bar{H}^\beta a^3 u(5+7u_c(1+5u_c)) + \bar{H}^\beta a^4(4u+4u_c+6uu_c+(-2+u(-5+4u(5+6u)))u_c^2) + \\
& + \bar{B}^\beta a^4(2u_c+u(2+u_c(5+24uu_c))) + \\
& + 2(4\bar{H}^\beta + a(3(1+a(7+2a(7+a))))\bar{B}^\beta + 4(8+a(2+a)(9+2a))\bar{H}^\beta - \\
& - 3(1+a(7+2a(7+a)))\bar{H}^\beta u_c) \left(\text{ArcTanh} \left[\frac{-1+2au}{\sqrt{1+4a}} \right] + \text{ArcTanh} \left[\frac{1-2au_c}{\sqrt{1+4a}} \right] \right) + \\
& + (1+a)(1+4a)^{3/2}(-4\bar{H}^\beta + a(-3\bar{B}^\beta + \bar{H}^\beta(-4+3u_c))) \times \\
& \times (\text{Log}[-1+u(-1+au)] - \text{Log}[-1+u_c(-1+au_c)]) \Big)
\end{aligned} \tag{131}$$

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